# Part III Quantum Information Theory: Example Sheet 2 v1.0 for example class 13th Nov, 2pm in MR14

2nd November

If anything is unclear or you think you've found a mistake, please email `wm266@cam.ac.uk`.

1. **State discrimination.** Suppose we know that a qubit is either in the pure state $\psi_0 = |\psi_0\rangle\langle\psi_0|$ where $|\psi_0\rangle := \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ or in the pure state $\psi_1 = |\psi_1\rangle\langle\psi_1|$ where $|\psi_1\rangle := \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$ (for some given $\theta \in [0, \pi/4]$) but we are completely unsure which it is, i.e. the state is $\psi_X$ where $X$ takes values in $\{0, 1\}$ and, $P_X(0) = P_X(1) = 1/2$.

   (a) If we measure a POVM with result $\hat{X}$ taking values in $\{0, 1\}$, what is the maximum success probability $\Pr(\hat{X} = X)$?

   (b) Argue that the maximum success probability in the previous part depends on the states only through the absolute value of their inner product $|\langle\psi_0|\psi_1\rangle|$.

   (c) Now suppose we are given $n$ qubits, either all prepared in state $\psi_0$ or all prepared in state $\psi_1$, so the state is $\psi_X^{\otimes n}$. If we measure a POVM on the whole $n$ qubit system with result $\hat{X}$ taking values in $\{0, 1\}$, what is the maximum success probability $\Pr(\hat{X} = X)$?

   (d) Now (for the original $n = 1$ case) suppose that we perform a POVM whose result $Y$ takes values in $\{0, 1, ?\}$, where the outcome ? means that we don't know which state the system is in.

   (a) Give elements $E(0), E(1)$ and $E(?)$ for the POVM such that

   $$\Pr(Y = 0|X = 1) = \Pr(Y = 1|X = 0) = 0$$

   (i.e. the measurement never gets the wrong state) and

   $$\Pr(?) = \cos(2\theta).$$

   (Hint: Start by considering the which forms of $E(0)$ and $E(1)$ are allowed by the constraints.) Try also to show that this the *smallest possible* value of $\Pr(?)$.

2. **Separable states and games.** Consider a game similar to the CHSH game where a referee sends a question $S$ to Alice and a question $T$ to Bob, and Alice and Bob respond to the referee with answers $X$ and $Y$ respectively. In the CHSH game the questions and answers were all bits, but now we make no assumption on the sets $\mathcal{A}_S$, $\mathcal{A}_T$, $\mathcal{A}_X$, $\mathcal{A}_Y$ except that they are all finite. There is some function $f : \mathcal{A}_S \times \mathcal{A}_T \times \mathcal{A}_X \times \mathcal{A}_Y \to \{0, 1\}$ and the players win the game iff $f(S, T, X, Y) = 1$.

   Suppose that, as in the quantum strategy for CHSH, Alice has a system $\mathsf{A}$ and Bob has a system $\mathsf{B}$; For each $s \in \mathcal{A}_S$ there is a POVM $E_s : \mathcal{A}_X \to \mathcal{L}(\mathcal{H}_\mathsf{A})$ and $X$ is the result of Alice measuring $E_S$ on $\mathsf{A}$; for each $t \in \mathcal{A}_T$ there is a POVM $F_t : \mathcal{A}_Y \to \mathcal{L}(\mathcal{H}_\mathsf{B})$, and $Y$ is the result of Bob measuring $F_T$ on $\mathsf{B}$. We make no assumption on the dimensions $d_\mathsf{A}$ and $d_\mathsf{B}$ except that they are finite.

   Show that, if the state of $\rho_{\mathsf{AB}}$ prior to the measurements is *separable*, then there is a local hidden variables model for how $X$ and $Y$ depend on $S$ and $T$, and therefore, the strategy will be no better than the best deterministic strategy.

3. **Checking separability.** Let $d_\mathsf{A} = d_\mathsf{B} = 2$ and let $\rho_{\mathsf{AB}}$ be the state whose matrix in the computational basis is

$$\rho_{\mathsf{AB}} = \frac{1}{5} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   Is $\rho_{\mathsf{AB}}$ separable?

4. **Generalising teleportation and dense coding.** On a $d$-dimensional Hilbert space, let us define operators

$$X := \sum_{0 \le j < d} |(j+1) \bmod d\rangle\langle j| \text{ and } Z := \sum_{0 \le j < d} \omega^j |j\rangle\langle j|,$$

   where $\omega := e^{2\pi i/d}$. By using properties of these operators (or otherwise) and assuming that Alice and Bob have systems $\mathsf{A}$ and $\mathsf{B}$, with $d_\mathsf{A} = d_\mathsf{B} = d$, initially in the state

$$\phi^+_{\mathsf{AB}} = |\phi^+\rangle\langle\phi^+|_{\mathsf{AB}}, \text{ where } |\phi^+\rangle_{\mathsf{AB}} = d^{-1/2} \sum_{i=0}^{d-1} |i\rangle_\mathsf{A} \otimes |i\rangle_\mathsf{B}$$

   describe:

   (a) A generalised dense coding protocol where Alice transmits a message $M$ which can take one of $d^2$ values to Bob (i.e. $|\mathcal{A}_M| = d^2$) by sending a $d$-dimensional system to Bob.

   (b) A generalised teleportation protocol where Alice transmits the state of a $d$-dimensional system to Bob by sending a (classical) message $M$ which takes one of $d^2$ values.

5. **Fidelity for states of qubits.** Suppose $\dim(\mathcal{H}_{\mathsf{Q}}) = 2$.

   (a) Show that, for any states $\rho$ and $\sigma$ of $\mathsf{Q}$, $F(\rho,\sigma)^2 = \mathrm{Tr}\rho\sigma + 2(\det(\rho)\det(\sigma))^{1/2}$.

   (b) For $v \in \mathbb{R}^3$ let $\rho[v] = \frac{1}{2}\left(\mathbb{1} + \sum_{i=1}^{3} v_i\sigma_i\right)$, where

   $$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

   This is Bloch ball representation of states of a qubit.
   Show that $F(\rho[u],\rho[v])^2 = \frac{1}{2}\left(1 + u\cdot v + [(1-u\cdot u)(1-v\cdot v)]^{1/2}\right)$.

6. **PPT entangled states.** For $0 \le j < 5$ let

   $$|\psi^{(j)}\rangle = \frac{1}{N}\left(\cos(2\pi j/5)|0\rangle + \sin(2\pi j/5)|1\rangle + h|2\rangle\right),$$

   where $h = \frac{1}{2}\sqrt{1+\sqrt{5}}$, and the constant $N$ is chosen so that these are unit vectors in a three dimensional complex Hilbert space. For $\dim(\mathsf{A}) = \dim(\mathsf{B}) = 3$, and $0 \le j < 5$ let

   $$\chi^{(j)}_{\mathsf{AB}} := |\chi^{(j)}\rangle\langle\chi^{(j)}|_{\mathsf{AB}},\ \text{where } |\chi^{(j)}\rangle_{\mathsf{AB}} := |\psi^{(j)}\rangle_{\mathsf{A}} \otimes |\psi^{(2j \bmod 5)}\rangle_{\mathsf{B}}.$$

   Consider the operator $M_{\mathsf{AB}} := \mathbb{1}_{\mathsf{AB}} - \sum_{0\le j<5} \chi^{(j)}_{\mathsf{AB}}$. Show that

   (a) $M_{\mathsf{AB}}$ is a projector (i.e. $M^\dagger_{\mathsf{AB}} M_{\mathsf{AB}} = M_{\mathsf{AB}}$) and $\mathbf{t}^{\mathsf{A}\leftarrow\mathsf{A}} M_{\mathsf{AB}} \ge 0$.

   (b) The kernel of $W_{\mathsf{AB}} := \sum_{0\le j<5} \chi^{(j)}_{\mathsf{AB}}$ contains no non-zero product vectors.

   (c) There are PPT states which are not separable.

7. **Trace distance and fidelity.** We denote the norm on operators $L \in \mathcal{L}(\mathcal{H})$ induced by the Hilbert-Schmidt inner product by $\|L\|_2 := \langle L, L\rangle^{1/2}$. This is known as the *Frobenius norm.*

   (a) Show that, if $\rho$ and $\sigma$ are both *pure* states, then the fidelity $F(\rho,\sigma)$ and trace distance $D(\rho,\sigma)$ are related by $F(\rho,\sigma) = \sqrt{1 - D(\rho,\sigma)^2}$.

   (b) Show that, for any two states $\rho$ and $\sigma$,

   $$F(\rho,\sigma) \le \sqrt{1 - D(\rho,\sigma)^2}.$$

   (c) In this part you can assume the following:

   **Lemma 1.** For any two positive operators $P \ge 0, Q \ge 0$,

   $$\|P - Q\|_1 \ge \|P^{1/2} - Q^{1/2}\|_2^2.$$

   Show that for any two states $\rho$ and $\sigma$,

   $$1 - D(\rho,\sigma) \le F(\rho,\sigma).$$

   (d) (Challenging question, optional) Prove the lemma.