7 Measurements

7.1 Instruments

Suppose we perform an operation $\mathcal{N}^{\tilde{\mathsf{X}}\mathsf{R}\leftarrow\mathsf{Q}}$ and then measure the system $\tilde{\mathsf{X}}$ in its computational basis (which we'll assume is labelled by the elements of some finite set $\mathcal{A}_X$) producing a result X. For each $x \in \mathcal{A}_X$, let $\mathcal{I}(x)^{\mathsf{R}\leftarrow\mathsf{Q}}$ be the linear map

$$\mathcal{I}(x)^{\mathsf{R}\leftarrow\mathsf{Q}} : \rho_{\mathsf{Q}} \mapsto \mathrm{Tr}_{\tilde{\mathsf{X}}} |x\rangle\langle x|_{\tilde{\mathsf{X}}} \otimes \mathbb{1}_{\mathsf{R}} (\mathcal{N}^{\tilde{\mathsf{X}}\mathsf{R}\leftarrow\mathsf{Q}} \rho_{\mathsf{Q}}) |x\rangle\langle x|_{\tilde{\mathsf{X}}} \otimes \mathbb{1}_{\mathsf{R}}. \tag{7.1}$$

Note that $\mathcal{I}(x)^{\mathsf{R}\leftarrow\mathsf{Q}}\rho_{\mathsf{Q}} = \langle x|_{\tilde{\mathsf{X}}} \otimes \mathbb{1}_{\mathsf{R}} (\mathcal{N}^{\tilde{\mathsf{X}}\mathsf{R}\leftarrow\mathsf{Q}} \rho_{\mathsf{Q}}) |x\rangle_{\tilde{\mathsf{X}}} \otimes \mathbb{1}_{\mathsf{R}}$. From the measurement postulate (for PVMs on density operators) we have

$$P_X(x) = \mathrm{Tr}_{\mathsf{R}} \mathcal{I}(x)^{\mathsf{R}\leftarrow\mathsf{Q}} \rho_{\mathsf{Q}} \tag{7.2}$$

and, given $X = x$ the state of $\tilde{\mathsf{X}}\mathsf{R}$ is $|x\rangle\langle x|_{\tilde{\mathsf{X}}} \otimes \mathcal{I}(x)^{\mathsf{R}\leftarrow\mathsf{Q}}\rho_{\mathsf{Q}}/P_X(x)$, so the state of R is

$$\frac{\mathcal{I}(x)^{\mathsf{R}\leftarrow\mathsf{Q}}\rho_{\mathsf{Q}}}{P_X(x)}. \tag{7.3}$$

The map $x \mapsto \mathcal{I}(x)^{\mathsf{R}\leftarrow\mathsf{Q}}$ is an *instrument.*

Definition 1. We can represent a measurement on Q whose result takes values in $\mathcal{A}_X$ and which leaves behind a system R by an **instrument** $\mathcal{I}$, which is a map $\mathcal{I} : \mathcal{A}_X \to \mathcal{L}(\mathcal{L}(\mathcal{H}_{\mathsf{Q}}), \mathcal{L}(\mathcal{H}_{\mathsf{R}}))$ such that

1. For all $x \in \mathcal{A}_X$, $\mathcal{I}(x)^{\mathsf{R}\leftarrow\mathsf{Q}}$ is completely positive;

2. $\sum_{x\in\mathcal{A}_X} \mathcal{I}(x)^{\mathsf{R}\leftarrow\mathsf{Q}}$ is trace preserving.

The distribution of the result X is given by (7.2) while the state of R immediately after the measurement, given $X = x$ is (7.3).

Any instrument $\mathcal{I} : \mathcal{A}_X \to \mathcal{L}(\mathcal{L}(\mathcal{H}_{\mathsf{Q}}), \mathcal{L}(\mathcal{H}_{\mathsf{R}}))$ can be implemented by following an operation $\mathcal{N}^{\tilde{\mathsf{X}}\mathsf{R}\leftarrow\mathsf{Q}}$ by a computational basis measurement on $\tilde{\mathsf{X}}$ as described above. A suitable choice of $\mathcal{N}^{\tilde{\mathsf{X}}\mathsf{R}\leftarrow\mathsf{Q}}$ is simply $\mathcal{N}^{\tilde{\mathsf{X}}\mathsf{R}\leftarrow\mathsf{Q}} = \sum_{x\in\mathcal{A}_X} |x\rangle\langle x|_{\tilde{\mathsf{X}}} \otimes \mathcal{I}(x)^{\mathsf{R}\leftarrow\mathsf{Q}}$. Note that

$$\mathcal{N}^{\tilde{\mathsf{X}}\mathsf{R}\leftarrow\mathsf{Q}} : \rho_{\mathsf{Q}} \mapsto \sum_{x\in\mathcal{A}_X} |x\rangle\langle x|_{\tilde{\mathsf{X}}} \otimes \mathcal{I}(x)^{\mathsf{R}\leftarrow\mathsf{Q}}\rho_{\mathsf{Q}}. \tag{7.4}$$

For example, the measurement of a PVM $E : \mathcal{A}_X \to \mathcal{L}(\mathcal{H}_{\mathsf{Q}})$ on a system Q is represented by the instrument $\mathcal{I} : \mathcal{A}_X \to \mathcal{L}(\mathcal{L}(\mathcal{H}_{\mathsf{Q}}), \mathcal{L}(\mathcal{H}_{\mathsf{Q}}))$ such that

$$\mathcal{I}(x)^{\mathsf{Q}\leftarrow\mathsf{Q}} : \rho_{\mathsf{Q}} \to E(x)_{\mathsf{Q}}\rho_{\mathsf{Q}}E(x)_{\mathsf{Q}}. \tag{7.5}$$

7.2 POVMs

If we are not interested in the post-measurement state, but only in the distribution of the result, then a measurement is completely specified by its associated POVM (positive-operator-valued measure). In fact, any linear function taking states to probability distributions can be represented by a POVM.

Definition 2. We can represent a measurement on a system Q, whose result takes values in $\mathcal{A}_X$, by a **POVM** E, which is a map $E : \mathcal{A}_X \to \mathcal{L}(\mathcal{H}_\mathsf{Q}) : x \mapsto E(x)$ such that

1. For all $x \in \mathcal{A}_X$, $E(x) \geq 0$ and
2. $\sum_{x\in\mathcal{A}_X} E(x) = \mathbb{1}_\mathsf{Q}$.

If the state of Q is ρ then $P_X(x) = \mathrm{Tr} E(x)\rho$.

We can write down the POVM for any instrument in terms of "adjoint maps".

Definition 3. On a space of operators $\mathcal{L}(\mathcal{H}_\mathsf{Q})$ we define the **Hilbert-Schmidt (HS) inner product** by $\langle L_\mathsf{Q}, J_\mathsf{Q}\rangle := \mathrm{Tr} L_\mathsf{Q}^\dagger J_\mathsf{Q}$. (Note: there is an exercise in the "Postulates…" handout which asks you to show that it is indeed an inner product).

Definition 4. The **adjoint map** of $\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_\mathsf{Q}), \mathcal{L}(\mathcal{H}_\mathsf{R}))$ is its hermitian adjoint w.r.t. the Hilbert-Schmidt inner product. That is, it is the unique map $\mathcal{M}^\dagger \in \mathcal{L}(\mathcal{L}(\mathcal{H}_\mathsf{R}), \mathcal{L}(\mathcal{H}_\mathsf{Q}))$ such that, for all $L_\mathsf{R} \in \mathcal{L}(\mathcal{H}_\mathsf{R})$, $J_\mathsf{Q} \in \mathcal{L}(\mathcal{H}_\mathsf{Q})$, $\langle L_\mathsf{R}, \mathcal{M} J_\mathsf{Q}\rangle = \langle \mathcal{M}^\dagger L_\mathsf{R}, J_\mathsf{Q}\rangle$.

Proposition 5. For any $\mathcal{M}, \mathcal{N} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_\mathsf{Q}), \mathcal{L}(\mathcal{H}_\mathsf{R}))$:

1. $(\alpha\mathcal{M} + \beta\mathcal{N})^\dagger = \alpha^*\mathcal{M}^\dagger + \beta^*\mathcal{N}^\dagger$ for all $\alpha, \beta \in \mathbb{C}$.
2. If $\mathcal{M} : J_\mathsf{Q} \mapsto \sum_j K_j J_\mathsf{Q} K_j^\dagger$, where $K_j \in \mathcal{L}(\mathcal{H}_\mathsf{Q}, \mathcal{H}_\mathsf{R})$, then $\mathcal{M}^\dagger : L_\mathsf{R} \mapsto \sum_j K_j^\dagger L_\mathsf{R} K_j$.
3. $\mathcal{M}$ is completely positive iff $\mathcal{M}^\dagger$ is completely positive.
4. $\mathcal{M}$ is trace preserving iff $\mathcal{M}^\dagger$ is **unital**, which means that $\mathcal{M}^\dagger \mathbb{1}_\mathsf{R} = \mathbb{1}_\mathsf{Q}$.
5. $(\mathcal{M}^\dagger)^\dagger = \mathcal{M}$.

Proof. (1) follows from the properties of inner products. (2) follows from definition of the HS inner product and the cyclicity and linearity of trace. (3) follows from (2) and the characterisation of CP maps as those with Kraus decompositions. (4) $\mathrm{Tr} J_\mathsf{Q} = \langle \mathbb{1}_\mathsf{Q}, J_\mathsf{Q}\rangle$ and $\mathrm{Tr}\mathcal{M} J_\mathsf{Q} = \langle \mathbb{1}_\mathsf{R}, \mathcal{M} J_\mathsf{Q}\rangle = \langle \mathcal{M}^\dagger \mathbb{1}_\mathsf{R}, J_\mathsf{Q}\rangle$. Therefore, $\mathrm{Tr} J_\mathsf{Q} = \mathrm{Tr}\mathcal{M} J_\mathsf{Q}$ for all $J_\mathsf{Q} \in \mathcal{L}(\mathcal{H}_\mathsf{Q})$ iff $\langle \mathbb{1}_\mathsf{Q}, J_\mathsf{Q}\rangle = \langle \mathcal{M}^\dagger \mathbb{1}_\mathsf{R}, J_\mathsf{Q}\rangle$ for all J_Q, and this last statement is equivalent to $\mathcal{M}^\dagger \mathbb{1}_\mathsf{R} = \mathbb{1}_\mathsf{Q}$. □

Proposition 6 (POVM for an instrument)**.** Given an instrument $\mathcal{I} : \mathcal{A}_X \to \mathcal{L}(\mathcal{L}(\mathcal{H}_\mathsf{Q}), \mathcal{L}(\mathcal{H}_\mathsf{R}))$ the corresponding POVM is the map $E : \mathcal{A}_X \to \mathcal{L}(\mathcal{H}_\mathsf{Q})$ with

$$E(x)_\mathsf{Q} = \mathcal{I}(x)^\dagger \mathbb{1}_\mathsf{R}. \tag{7.6}$$

Proof. Since $\mathcal{I}(x)$ is CP, its adjoint is too, and since $\mathbb{1}_\mathsf{R} \geq 0$, $E(x)_\mathsf{Q} \geq 0$. $\mathcal{N} := \sum_{x\in\mathcal{A}_X} \mathcal{I}(x)$ is trace preserving, so $\mathcal{N}^\dagger$ is unital, and $\sum_{x\in\mathcal{A}_X} E(x)_\mathsf{Q} = \sum_{x\in\mathcal{A}_X} \mathcal{I}(x)^\dagger \mathbb{1}_\mathsf{R} = \mathcal{N}^\dagger \mathbb{1}_\mathsf{R} = \mathbb{1}_\mathsf{Q}$. Therefore, (7.6) indeed defines a POVM, and

$$P_X(x) = \mathrm{Tr}_\mathsf{R}\mathcal{I}(x)\rho_\mathsf{Q} = \langle \mathbb{1}_\mathsf{R}, \mathcal{I}(x)\rho_\mathsf{Q}\rangle = \langle \mathcal{I}(x)^\dagger \mathbb{1}_\mathsf{R}, \rho_\mathsf{Q}\rangle = \mathrm{Tr} E(x)_\mathsf{Q}\rho_\mathsf{Q}$$

so its result has the same distribution as the instrument, for any state ρ_Q. □

Proposition 7. For any POVM $E : \mathcal{A}_X \to \mathcal{L}(\mathcal{H}_\mathsf{Q})$ we can find some (non-unique) instrument with that POVM.

Proof. It is easy to check that setting

$$\mathcal{I}(x)^{\mathsf{Q}\leftarrow\mathsf{Q}} : \rho_\mathsf{Q} \mapsto E(x)_\mathsf{Q}^{1/2} \rho_\mathsf{Q} E(x)_\mathsf{Q}^{1/2}$$

defines an instrument $\mathcal{I} : \mathcal{A}_X \to \mathcal{L}(\mathcal{L}(\mathcal{H}_\mathsf{Q}), \mathcal{L}(\mathcal{H}_\mathsf{Q}))$ with the given POVM. □

The POVM for a PVM $E : \mathcal{A}_X \to \mathcal{L}(\mathcal{H}_\mathsf{Q})$ is simply E itself. It is important to note that when we only give a POVM for a measurement, this does not give us enough information to determine the post-measurement states.

7.3 Summary of measurement representations

We have now seen all the ways of representing a measurement that we will encounter in this course: PVMs, Instruments and POVMs.

Name	Form	$P_X(x)$	State given $X = x$
PVM	$E : \mathcal{A}_X \to \mathcal{L}(\mathcal{H}_\mathsf{Q})$, where $E(x)^\dagger E(x') = \delta_{x'x} E(x)$, and $\sum_{x\in\mathcal{A}_X} E(x) = \mathbb{1}$.	$\mathrm{Tr} E(x)\rho$	$E(x)\rho E(x)/P_X(x)$
Instrument	$\mathcal{I} : \mathcal{A}_X \to \mathcal{L}(\mathcal{L}(\mathcal{H}_\mathsf{Q}), \mathcal{L}(\mathcal{H}_\mathsf{R}))$, where $\mathcal{I}(x)$ is CP for all $x \in \mathcal{A}_X$, and $\sum_{x\in\mathcal{A}_X} \mathcal{I}(x)$ is TP.	$\mathrm{Tr}_\mathsf{R} \mathcal{I}(x)\rho$	$\mathcal{I}(x)\rho/P_X(x)$
POVM	$E : \mathcal{A}_X \to \mathcal{L}(\mathcal{H}_\mathsf{Q})$, where $E(x) \geq 0$ for all $x \in \mathcal{A}_X$, and $\sum_{x\in\mathcal{A}_X} E(x) = \mathbb{1}$.	$\mathrm{Tr} E(x)\rho$	unspecified

Table 7.1: For measurements on system Q in state ρ_Q with result X we give the general form of the representation, the distribution P_X of the result, and the state immediately after the measurement when $X = x$.