4.6 A closer look at the quantum strategy

Let's analyse the quantum strategy for the CHSH game described in section 4.4.1 in a sightly different way. Basic probability tells us that

$$P_{XY|ST}(x,y|s,t) = P_{Y|XST}(y|x,s,t)P_{X|ST}(x|s,t). \tag{4.19}$$

Suppose that Alice measures first. Then, the measurement postulate says that

$$P_{X|ST}(x|s,t) = P_{X|S}(x|s) = \langle\phi^+|E_s(x)_{\mathsf{A}} \otimes \mathbb{1}_{\mathsf{B}}|\phi^+\rangle = \|E_s(x)_{\mathsf{A}} \otimes \mathbb{1}_{\mathsf{B}}|\phi^+\rangle\|^2, \tag{4.20}$$

and immediately after Alice's measurement, given $S = s$ and $X = x$, the state of AB is

$$E_s(x)_{\mathsf{A}} \otimes \mathbb{1}_{\mathsf{B}}|\phi^+\rangle / \|E_s(x)_{\mathsf{A}} \otimes \mathbb{1}_{\mathsf{B}}|\phi^+\rangle\|. \tag{4.21}$$

The projectors $E_s(x)$ are rank-1 for all s, x. For an arbitrary rank-1 projector, which we can write as $|\psi\rangle_{\mathsf{A}}$ where $\langle\psi|\psi\rangle = 1$, we have

$$|\psi\rangle\langle\psi|_{\mathsf{A}} \otimes \mathbb{1}_{\mathsf{B}}|\phi^+\rangle_{\mathsf{AB}} = |\psi\rangle_{\mathsf{A}}(\langle\psi|_{\mathsf{A}} \otimes \mathbb{1}_{\mathsf{B}})(|0\rangle_{\mathsf{A}} \otimes |0\rangle_{\mathsf{B}} + |1\rangle_{\mathsf{A}} \otimes |1\rangle_{\mathsf{B}})/\sqrt{2} \tag{4.22}$$

$$= |\psi\rangle_{\mathsf{A}}(\langle\psi|0\rangle|0\rangle_{\mathsf{B}} + \langle\psi|0\rangle|0\rangle_{\mathsf{B}})/\sqrt{2} \tag{4.23}$$

$$= \frac{1}{\sqrt{2}}|\psi\rangle_{\mathsf{A}} \otimes |\psi\rangle_{\mathsf{B}}^*, \tag{4.24}$$

where $|\psi\rangle_{\mathsf{B}} = \mathbb{1}_{\mathsf{B}\leftarrow\mathsf{A}}|\psi\rangle_{\mathsf{A}}$. The linear map $\mathbb{1}_{\mathsf{B}\leftarrow\mathsf{A}} \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{B}})$ is the map which takes the computational basis vectors of $\mathcal{H}_{\mathsf{A}}$ to the corresponding computational basis vectors of $\mathcal{H}_{\mathsf{B}}$:

$$\forall i \in \{0, \ldots, d_{\mathsf{A}} - 1\} : \; \mathbb{1}_{\mathsf{B}\leftarrow\mathsf{A}}|i\rangle_{\mathsf{A}} = |i\rangle_{\mathsf{B}}.$$

In (4.24) we used the fact that we have declared computational basis vectors to be real (i.e. $|i\rangle^* = |i\rangle$) and the general fact $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$ to compute

$$|\psi\rangle^* = (|0\rangle\langle 0|\psi\rangle + |1\rangle\langle 1|\psi\rangle)^* = |0\rangle\langle\psi|0\rangle + |1\rangle\langle\psi|1\rangle.$$

Note that, for any unit vector $|\psi\rangle \in \mathcal{H}_{\mathsf{A}}$, $\||\psi\rangle\langle\psi|_{\mathsf{A}} \otimes \mathbb{1}_{\mathsf{B}}|\phi^+\rangle_{\mathsf{AB}}\|^2 = 1/2$. Therefore, when the state of AB is $|\phi^+\rangle_{\mathsf{AB}}$, for *any* basis measurement (i.e. any PVM with rank-1 projectors), the two values the result can take have the same probability! In particular,

$$\forall x, s : P_{X|S}(x|s) = 1/2 \tag{4.25}$$

and, immediately after Alice's measurement, given $X = x, S = s$, the state of B is $|\beta_{(s,x)}\rangle := |\eta[\pi(x + s/2)]\rangle^* = |\eta[-\pi(x + s/2)]\rangle$, or, as a projector

$$\beta_{(s,x)} = |\beta_{(s,x)}\rangle\langle\beta_{(s,x)}| = \eta[-\pi(x + s/2)]. \tag{4.26}$$

In figure 4.1, we plot these four states on the Bloch sphere for B. Since Bob's PVMs have rank-1 projectors, we can plot these projectors on the Bloch sphere, also.

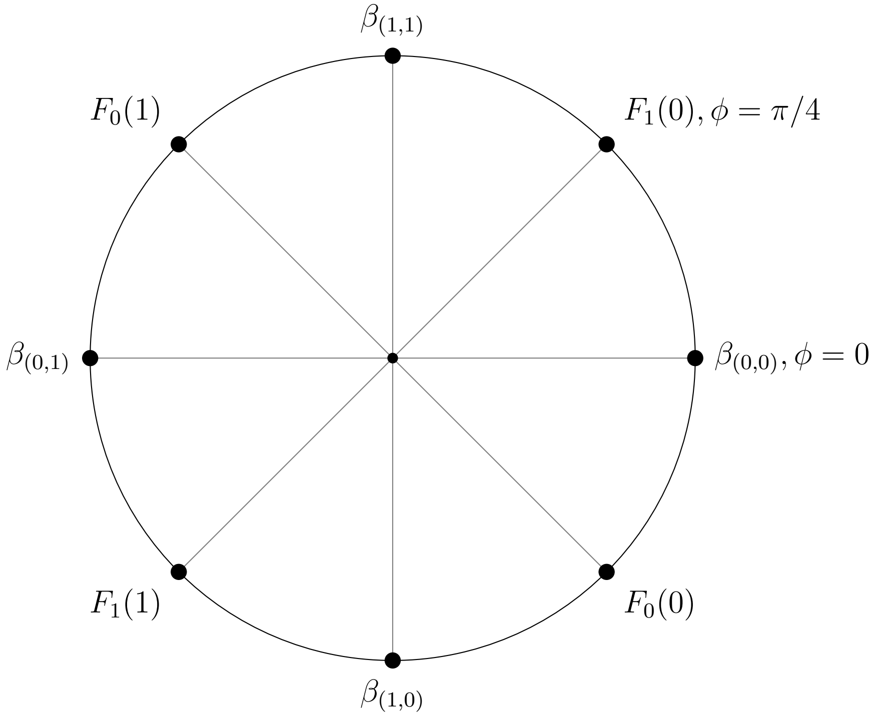

Figure 4.1: The circle is the equator of the Bloch sphere, viewed from above. On this we have plotted the states $|\beta_{(s,x)}\rangle$ of Bob's qubit after Alice's measurement, when $S = s$ (the question S she got had value s) and $X = x$ (her measurement result X had value x). We have also plotted the rank-1 projectors $F_t(y)$ corresponding to Bob's result having value y when he measures F_t (i.e. when the question T he receives has value t).

Now, given $S = s, X = x$ and $T = t$, the probability distribution for Bob's measurement result is

$$\begin{aligned} P_{Y|XST}(y|x,s,t) =& \langle\beta_{(s,x)}|F_t(y)|\beta_{(s,x)}\rangle = \mathrm{Tr} F_t(y)\beta_{(s,x)} && (4.27)\\ =& \mathrm{Tr}\eta[\pi(t/2 - y - 1/4)]\eta[\pi(x + s/2)] \text{ so, by (4.25),} && (4.28)\\ P_{XY|ST}(x,y|s,t) =& P_{Y|XST}(y|x,s,t)P_{X|ST}(x|s,t) && (4.29)\\ =& \frac{1}{2}\mathrm{Tr}\eta[\pi(t/2 - y - 1/4)]\eta[\pi(x + s/2)]. && (4.30) \end{aligned}$$

If the state of B is $\eta[\phi]$, then the probability of an obtaining a measurement result which is associated to the projector $\eta[\phi']$ is

$$\begin{aligned} \mathrm{Tr}\eta[\phi']\eta[\phi] = |\langle\eta[\phi']|\eta[\phi]\rangle|^2 =& \frac{1}{4}|1 + e^{i(\phi-\phi')}|^2 = \frac{1}{4}(1 + \mathrm{Re}[e^{i(\phi-\phi')}]) && (4.31)\\ =& \frac{1}{2}(1 + \cos(\phi - \phi')) = \cos^2\left(\frac{\phi - \phi'}{2}\right). && (4.32) \end{aligned}$$

This is zero when $|\phi - \phi'| = \pi$, and increases as $|\phi - \phi'|$ goes to zero. Using (4.30) and (4.32), and looking at the figure, we see that

$$\begin{aligned} \Pr(\mathbf{WIN}|ST = 0) =& \Pr(X = Y|ST = 0) = \cos^2(\pi/8) = \frac{1}{2}(1 + \frac{1}{\sqrt{2}}), \text{ and}\\ \Pr(\mathbf{WIN}|ST = 1) =& \Pr(X \neq Y|ST = 1) = \cos^2(\pi/8). \end{aligned}$$

5 Density operators

5.1 Uncertainty about the state vector

Suppose that we know that the state vector of a system Q belongs to some set $\{|\psi_x\rangle : x \in \mathcal{A}_X\}$ but that we aren't sure which. We represent the identity of the state as a random variable X with values in $\mathcal{A}_X$ and distribution P_X. Now, suppose that we perform a measurement with result Y (an RV taking values in $\mathcal{A}_Y$), and PVM E. Then $P_{Y|X}(y|x) = \langle\psi_x|E(y)|\psi_x\rangle = \mathrm{Tr}E(y)|\psi_x\rangle\langle\psi_x|$, and

$$P_Y(y) = \sum_{x\in\mathcal{A}_X} P_{Y|X}(y|x)P_X(x) = \mathrm{Tr}E(y)\rho$$

where $\rho = \sum_x P_X(x)|\psi_x\rangle\langle\psi_x| \in \mathcal{L}(\mathcal{H}_\mathsf{Q})$. So, the distribution, P_Y, of the measurement result Y does not depend on the particular **ensemble** $\{(P_X(x), |\psi_x\rangle\langle\psi_x|) : x \in \mathcal{A}_X\}$ but only on the **ensemble average** ρ. It is easily verified that any ensemble average ρ satisfies $\rho \geq 0$ and $\mathrm{Tr}\rho = 1$. Such an operator is called a **density operator**:

Definition 1 (Density operator)**.** A density operator on $\mathcal{H}_\mathsf{Q}$ an operator $\rho \in \mathrm{Herm}(\mathcal{H}_\mathsf{Q})$ with $\rho \geq 0$ (meaning ρ is a positive operator, see section 2.2.9) and $\mathrm{Tr}\rho = 1$. We will denote the set of all density operators on $\mathcal{H}_\mathsf{Q}$ by $\mathcal{D}(\mathcal{H}_\mathsf{Q})$.

For any state vector $|\psi\rangle$, there is a corresponding density operator $|\psi\rangle\langle\psi|$ which determines the state vector up to a (physically irrelevant) global phase. From now on, when we talk about a **state** of Q, we generally mean a density operator on $\mathcal{H}_\mathsf{Q}$. A **pure state** is a state of the form $|\psi\rangle\langle\psi|$ for some state vector $|\psi\rangle$. A state which is not pure is called **mixed**. Equivalently, a state is pure if its density operator is rank-1. The **maximally mixed state** of ρ is $\mathbb{1}_\mathsf{Q}/d_\mathsf{Q}$ (where $\mathbb{1}_\mathsf{Q} = \sum_{0\leq j<d_\mathsf{Q}} |j\rangle\langle j|_\mathsf{Q}$ denotes the identity operator on $\mathcal{H}_\mathsf{Q}$).

Proposition 2. Any density operator is the ensemble average of some ensemble of pure states.

Proof. Let ρ be any density operator on $\mathcal{H}_\mathsf{Q}$. ρ has an eigendecomposition $\sum_{0\leq j<d_\mathsf{Q}} \lambda_j|\alpha_j\rangle\langle\alpha_j|$[1] and since $\rho \geq 0$ iff $\lambda_j \geq 0$ for all j and $\mathrm{Tr}\rho = 1$ iff $\sum_{0\leq j<d_\mathsf{Q}} \lambda_j = 1$, the eigenvalues correspond to a probability distribution (on $\{0, \ldots, d_\mathsf{Q} - 1\}$, say), and ρ is the ensemble average of $\{(\lambda_j, |\alpha_j\rangle\langle\alpha_j|) : 0 \leq j < d_\mathsf{Q}\}$. □

We have shown that the set $\mathcal{D}(\mathcal{H}_\mathsf{Q})$ of density operators on $\mathcal{H}_\mathsf{Q}$ is the convex hull of the set of pure states (as projectors) $\{|\psi\rangle\langle\psi| : |\psi\rangle \in \mathcal{H}_\mathsf{Q}, \langle\psi|\psi\rangle = 1\}$.

5.2 Measuring PVMs when the state is mixed

Let's derive a "measurement postulate" for the measurement of PVMs when our state is described by a density operator.

[1] By this I mean that, for each i, j, $\langle\alpha_i|\alpha_j\rangle = \delta_{ij}$, $\rho|\alpha_i\rangle = \lambda_i|\alpha_i\rangle$ and $\rho = \sum_{0\leq j<d_\mathsf{Q}} \lambda_j|\alpha_j\rangle\langle\alpha_j|$.

Proposition 3. If the state of a system Q is ρ, where ρ is a density operator in $\mathcal{L}(\mathcal{H}_\mathsf{Q})$ and a PVM $E : \mathcal{A}_Y \to \mathcal{L}(\mathcal{H}_\mathsf{Q})$ is measured with result Y, then

1. $P_Y(y) = \mathrm{Tr}E(y)\rho$ and

2. Immediately after the measurement, if $Y = y$, then the state of Q is $\frac{E(y)\rho E(y)}{\mathrm{Tr}E(y)\rho}$.

Proof. We already established (1) in the previous section, so it remains to derive (2). Suppose that we have assigned ρ to Q because we know that Q has pure state $|\psi_x\rangle$ with probability $P_X(x)$, and

$$\sum_{x\in\mathcal{A}_X} P_X(x)|\psi_x\rangle\langle\psi_x| = \rho. \tag{5.1}$$

Using the measurement postulate, we know that for a measurement of E, the distribution of Y given X can is

$$P_{Y|X}(y|x) = \mathrm{Tr}E(y)|\psi_x\rangle\langle\psi_x| \tag{5.2}$$

(note that this is only defined when $P_X(x) > 0$) and immediately after the measurement, if $X = x$ and $Y = y$ then the state of Q is

$$|\psi_{x,y}\rangle = \frac{E(y)|\psi_x\rangle}{\sqrt{P_{Y|X}(y|x)}} \tag{5.3}$$

which is defined only when $P_{Y|X}(y|x) > 0$ and $P_X(x) > 0$, or equivalently, when $P_{XY}(x,y) > 0$. Conditional on $Y = y$ (we can assume $P_Y(y) > 0$) the state of Q is $|\psi_{x,y}\rangle$ with probability $P_{X|Y}(x|y)$, so we must assign to Q the state (density operator)

$$\sum_x P_{X|Y}(x|y)|\psi_{x,y}\rangle\langle\psi_{x,y}| = \sum_x P_{X|Y}(x|y)\frac{E(y)|\psi_x\rangle\langle\psi_x|E(y)}{P_{Y|X}(y|x)} \tag{5.4}$$

$$= \frac{E(y)\left(\sum_x P_X(x)|\psi_x\rangle\langle\psi_x|\right)E(y)}{P_Y(y)} = \frac{E(y)\rho E(y)}{\mathrm{Tr}E(y)\rho}. \tag{5.5}$$

Here we are summing only over x such that $P_{X|Y}(x|y) > 0$, and since we are assuming $P_Y(y) > 0$, we have $P_{XY}(x,y) > 0$ for all terms in the sum. We have used the fact that $\frac{P_{X|Y}(x|y)}{P_{Y|X}(y|x)} = \frac{P_{XY}(x,y)P_X(x)}{P_{XY}(x,y)P_Y(y)}$ and (5.1). Note that our derivation did not depend on the specific pure state ensemble $\{(P_X(x), |\psi_x\rangle\langle\psi_x|) : x \in \mathcal{A}_X\}$. □

5.3 Storing classical information in quantum systems

Evidently it is possible to reliably store information in quantum systems. If we accept that the universe is quantum mechanical, then *all* reliable information storage methods are evidence of this!

One way this can be done is to use an orthogonal basis of pure states to represent the value we wish to store. For example, we could store a classical bit in a qubit by using $|0\rangle$ to represent 0 and $|1\rangle$ represent 1. To retrieve the bit, one would measure the qubit in the computational basis. It would work just as well to use $|+\rangle$ and $|-\rangle$ to represent the two possible values, because they are orthogonal and, therefore, a measurement in the $\{|+\rangle, |-\rangle\}$ basis always gives the correct value for the stored bit.

More generally, given a value in some finite set $\mathcal{A}_X$, we can store it in a quantum system $\tilde{\mathsf{X}}$ with $d_{\tilde{\mathsf{X}}} = |\mathcal{A}_X|$. For convenience, we label the computational basis of the system $\tilde{\mathsf{X}}$ by the elements of $\mathcal{A}_X$ rather than numbers, and we store $x \in \mathcal{A}_X$ as $|x\rangle_{\tilde{\mathsf{X}}}$.

Suppose we store a particular value $x \in \mathcal{A}_X$ in $\tilde{\mathsf{X}}$ in this way. If we measure the PVM

$$C : \mathcal{A}_X \to \mathcal{L}(\mathcal{H}_{\tilde{\mathsf{X}}}) : x \mapsto |x\rangle\langle x|$$

(this is measurement in the computational basis) and call the result X', then

$$P_{X'}(x') = \mathrm{Tr} C(x')|x\rangle\langle x| = |\langle x'|x\rangle|^2 = \delta_{x'x}.$$

So, $X' = x$ with certainty: the storage is perfectly reliable. Furthermore, after measuring E, the state of $\tilde{\mathsf{X}}$ is certainly

$$C(x)|x\rangle\langle x|C(x) = |x\rangle\langle x|.$$

That is, the state (and hence, the value stored) has not been disturbed by the measurement.

If we say, without qualification, that a random variable X is stored in system $\tilde{\mathsf{X}}$, then we mean that the value of X has been stored in the way just described, and assume that the computational basis of $\tilde{\mathsf{X}}$ is labeled by $\mathcal{A}_X$. If X is stored in $\tilde{\mathsf{X}}$ then the state $\rho_{\tilde{\mathsf{X}}}$ of $\tilde{\mathsf{X}}$ is determined by the distribution P_X

$$\rho_{\tilde{\mathsf{X}}} = \sum_{x \in \mathcal{A}_X} P_X(x)|x\rangle\langle x|. \tag{5.6}$$

If I measure C on $\tilde{\mathsf{X}}$ obtaining result X' then, unsurprisingly,

$$P_{X'}(x') = \mathrm{Tr} C(x')\rho_{\tilde{\mathsf{X}}} = P_X(x').$$

What happens to the state of $\tilde{\mathsf{X}}$? Well, the previous section tells us that, if $X' = x'$ the state we should assign to $\tilde{\mathsf{X}}$ after the measurement is

$$\frac{C(x')\rho_{\tilde{\mathsf{X}}}C(x')}{\mathrm{Tr} C(x')\rho_{\tilde{\mathsf{X}}}} = |x'\rangle\langle x'|.$$

In this case, the form of the new state just reflects the fact that we have learnt that $X = x'$.

5.3.1 Copying classical information

There is nothing to stop us from building a device that measures C on $\tilde{\mathsf{X}}$ and the stores the result X' in another system $\tilde{\mathsf{X}}'$. The effect of this is to make a *copy* (or *clone*) the value stored in $\tilde{\mathsf{X}}$. If $\tilde{\mathsf{X}}$ has state $\rho_{\tilde{\mathsf{X}}}$, as in (5.6), before the device is applied, then afterwards we know that the state of $\tilde{\mathsf{X}}\tilde{\mathsf{X}}'$ must be

$$\sum_x P_X(x)|x\rangle\langle x|_{\tilde{\mathsf{X}}} \otimes |x\rangle\langle x|_{\tilde{\mathsf{X}}'}. \tag{5.7}$$

The form of the state reflects the fact that $\Pr(X' = X) = 1$: the original data and its copy are perfectly correlated.

5.4 States of subsystems

5.4.1 Partial trace

Recall that in section 2.2.7 we defined the trace $\mathrm{Tr}M$ of an operator $M \in \mathcal{L}(\mathcal{H})$

$$\mathrm{Tr}M := \sum_{0\le i<\dim(\mathcal{H})} \langle i|M|i\rangle.$$

Proposition 4. If Tr_R is a linear map from $\mathcal{L}(\mathcal{H}_\mathsf{Q} \otimes \mathcal{H}_\mathsf{R})$ to $\mathcal{L}(\mathcal{H}_\mathsf{Q})$, then the following statements are equivalent

1. $\forall M_\mathsf{QR} \in \mathcal{L}(\mathcal{H}_\mathsf{Q} \otimes \mathcal{H}_\mathsf{R}), J_\mathsf{Q} \in \mathcal{L}(\mathcal{H}_\mathsf{Q}):\ \mathrm{Tr}J_\mathsf{Q} \otimes \mathbb{1}_\mathsf{R} M_\mathsf{QR} = \mathrm{Tr}J_\mathsf{Q}\mathrm{Tr}_\mathsf{R}M_\mathsf{QR}$;
2. $\forall K_\mathsf{Q} \in \mathcal{L}(\mathcal{H}_\mathsf{Q}), L_\mathsf{R} \in \mathcal{L}(\mathcal{H}_\mathsf{R}):\ \mathrm{Tr}_\mathsf{R}K_\mathsf{Q} \otimes L_\mathsf{R} = K_\mathsf{Q}(\mathrm{Tr}L_\mathsf{R})$;
3. $\mathrm{Tr}_\mathsf{R}M_\mathsf{QR} = \sum_{0\le r<d_\mathsf{R}}(\mathbb{1}_\mathsf{Q} \otimes \langle r|_\mathsf{R})M_\mathsf{QR}(\mathbb{1}_\mathsf{Q} \otimes |r\rangle_\mathsf{R})$;

and there a unique map Tr_R which satisfies them.

In the third statement, note that $\mathbb{1}_\mathsf{Q} \otimes \langle r|_\mathsf{R}$ is a linear map from $\mathcal{H}_\mathsf{Q} \otimes \mathcal{H}_\mathsf{R}$ to $\mathcal{H}_\mathsf{Q}$, while $\mathbb{1}_\mathsf{Q} \otimes |r\rangle_\mathsf{R}$ is a linear map from $\mathcal{H}_\mathsf{Q}$ to $\mathcal{H}_\mathsf{Q} \otimes \mathcal{H}_\mathsf{R}$. We can write them in terms of computational basis bras and kets as

$$\mathbb{1}_\mathsf{Q} \otimes \langle r|_\mathsf{R} = \sum_{0\le q<d_\mathsf{Q}} |q\rangle_\mathsf{Q}\langle q|_\mathsf{Q} \otimes \langle r|_\mathsf{R} \text{ and } \mathbb{1}_\mathsf{Q} \otimes |r\rangle_\mathsf{R} = \sum_{0\le q<d_\mathsf{Q}} |q\rangle_\mathsf{Q} \otimes |r\rangle_\mathsf{R}\langle q|_\mathsf{Q}. \tag{5.8}$$

Definition 5 (Partial trace)**.** The unique linear map Tr_R from $\mathcal{L}(\mathcal{H}_\mathsf{Q} \otimes \mathcal{H}_\mathsf{R})$ to $\mathcal{L}(\mathcal{H}_\mathsf{Q})$, which satisfies the three, equivalent, statements in the preceding proposition is called the "partial trace over R".

Example 6. Consider the state $|\phi^+\rangle_\mathsf{AB}$ we used in our quantum strategy for the CHSH game. Using the second definition of partial trace from Prop. 4, we compute

$$\mathrm{Tr}_\mathsf{B}|\phi^+\rangle\langle\phi^+|_\mathsf{AB} = \mathrm{Tr}_\mathsf{B}\frac{1}{2}\left(\sum_{i,j=0}^{1} |i\rangle_\mathsf{A} \otimes |i\rangle_\mathsf{B}\langle j|_\mathsf{A} \otimes \langle j|_\mathsf{B}\right) \tag{5.9}$$

$$= \frac{1}{2}\mathrm{Tr}_\mathsf{B}\left(\sum_{i,j=0}^{1} |i\rangle\langle j|_\mathsf{A} \otimes |i\rangle\langle j|_\mathsf{B}\right) \tag{5.10}$$

$$= \frac{1}{2}\sum_{i,j=0}^{1} |i\rangle\langle j|_\mathsf{A}(\mathrm{Tr}|i\rangle\langle j|_\mathsf{B}) \tag{5.11}$$

$$= \frac{1}{2}\sum_{i,j=0}^{1} |i\rangle\langle j|_\mathsf{A}\langle i|j\rangle = \frac{1}{2}\sum_{i=0}^{1} |i\rangle\langle i|_\mathsf{A}. \tag{5.12}$$

So, $\mathrm{Tr}_\mathsf{B}|\phi^+\rangle\langle\phi^+|_\mathsf{AB} = \mathbb{1}_\mathsf{A}/2$ is the maximally mixed state of A.

Example 7. Suppose $d_\mathsf{Q} = d_\mathsf{R} = 2$, then we can write out any M_QR in the computational basis $M_\mathsf{QR} = \sum_{i,j,k,l=0}^{1} m_{ij,kl}|i\rangle_\mathsf{Q} \otimes |j\rangle_\mathsf{R}\langle k|_\mathsf{Q} \otimes \langle l|_\mathsf{R}$. The matrix representation of this is

$$M_\mathsf{QR} = \begin{pmatrix} m_{00,00} & m_{00,01} & m_{00,10} & m_{00,11} \\ m_{01,00} & m_{01,01} & m_{01,10} & m_{01,11} \\ m_{10,00} & m_{10,01} & m_{10,10} & m_{10,11} \\ m_{11,00} & m_{11,01} & m_{11,10} & m_{11,11} \end{pmatrix} \tag{5.13}$$

then

$$\mathrm{Tr}_{\mathsf{R}} M_{\mathsf{QR}} = \begin{pmatrix} m_{00,00} + m_{01,01} & m_{00,10} + m_{01,11} \\ m_{10,00} + m_{11,01} & m_{10,10} + m_{11,11} \end{pmatrix}, \text{ and}$$
$$\mathrm{Tr}_{\mathsf{Q}} M_{\mathsf{QR}} = \begin{pmatrix} m_{00,00} + m_{10,10} & m_{00,01} + m_{10,11} \\ m_{10,00} + m_{11,01} & m_{01,01} + m_{11,11} \end{pmatrix}.$$

5.4.2 States of subsystems

Suppose that we have a composite system QR which we know to have state vector $|\psi\rangle_{\mathsf{QR}}$. Unless $|\psi\rangle_{\mathsf{QR}}$ is a product vector, we cannot represent the state of subsystem Q (or R) by a state vector. *What is the state of* Q?

To be general, let's suppose the state of QR is any density operator ρ_{QR} (which could be the pure state $|\psi\rangle\langle\psi|_{\mathsf{QR}}$, for example). Consider measuring a PVM E on Q with result Y. Then, from our discussion of measurements on composite systems in section 4.3.6 of Handout 2, and from section 5.2 we know that

$$\Pr(Y = y) = \mathrm{Tr} E(y)_{\mathsf{Q}} \otimes \mathbb{1}_{\mathsf{R}} \rho_{\mathsf{QR}} = \mathrm{Tr} E(y)_{\mathsf{Q}} \rho_{\mathsf{Q}} \tag{5.14}$$

where $\rho_{\mathsf{Q}} = \mathrm{Tr}_{\mathsf{R}} \rho_{\mathsf{QR}}$ is given by the partial trace of ρ_{QR} over R. (Applying the map Tr_{R} is sometimes called "tracing out" R.) It is easy to check that ρ_{Q} is a density operator on $\mathcal{H}_{\mathsf{Q}}$, since it determines the probabilities of any PVM we might measure on the system Q, ρ_{Q} must be the state that we assign to Q.

Proposition 8 (♣♣ Prove this.). If ρ_{QR} is a density operator, then $\rho_{\mathsf{Q}} = \mathrm{Tr}_{\mathsf{R}} \rho_{\mathsf{QR}}$ is a density operator.

5.5 Extensions and Purifications

Definition 9. If a state η_{QR} of QR satisfies $\mathrm{Tr}_{\mathsf{R}} \eta_{\mathsf{QR}} = \rho_{\mathsf{Q}}$ then we say that η_{QR} is an **extension** of the state ρ_{Q} to QR. An extension of ρ_{Q} which is pure is called a **purification** of ρ_{Q}.

Proposition 10. Suppose that we have a system Q with state ρ_{Q}. It is always possible to come up with a system R (we are not saying this system really exists) and a pure state η_{QR} such that $\mathrm{Tr}_{\mathsf{R}} \eta_{\mathsf{QR}} = \rho_{\mathsf{Q}}$ - that is, η_{QR} is a purification of ρ_{Q}.

Proof. We know there is at least one ensemble of pure states $\{(P(x), |\psi_x\rangle\langle\psi_x|_{\mathsf{Q}}) : 0 \leq x < k\}$ such that $\rho_{\mathsf{Q}} = \sum_x P(x)|\psi_x\rangle\langle\psi_x|_{\mathsf{Q}}$. Let R be a k-dimensional system, and let $|\psi\rangle_{\mathsf{QR}} = \sum_{0 \leq x < k} \sqrt{P(x)}|\psi_x\rangle_{\mathsf{Q}} \otimes |x\rangle_{\mathsf{R}}$. Then, using $\mathrm{Tr}_{\mathsf{R}}|x\rangle\langle x'|_{\mathsf{R}} = \langle x'|x\rangle = \delta_{x',x}$, the state of Q is

$$\mathrm{Tr}_{\mathsf{R}}|\psi\rangle\langle\psi|_{\mathsf{QR}} = \sum_{0 \leq x, x' < k} \sqrt{P(x)P(x')}|\psi_x\rangle\langle\psi_{x'}|_{\mathsf{Q}} \langle x'|x\rangle_{\mathsf{R}} = \rho_{\mathsf{Q}} \tag{5.15}$$

as required. □

We should take care to note what Prop. 10 does *not* say. Suppose we have a system AB in some state ρ_{AB}. While we can certainly find a purification η_{ABR} (we are just taking $\mathsf{Q} = \mathsf{AB}$ in the Prop. 10) it is not necessarily possible to find any extension η_{ABR} such that the state $\eta_{\mathsf{AR}} = \mathrm{Tr}_{\mathsf{B}} \eta_{\mathsf{ABR}}$ of AR *alone* is pure. For example, if A and B are qubits and the state of AB is $|\phi^+\rangle$ (the state we used in the CHSH game) then there is no such extension.

5.6 Quick exercises for the first examples class ♣♣

1. Suppose that we know that, initially, a qubit Q is in the state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. We then learn that someone has performed a computational basis measurement on Q, but we do not learn the result of this measurement. What state should we assign to Q, now?

2. If the state of a pair of qubits AB is $\sum_{i,j=0}^{1} \alpha_{ij} |i\rangle \otimes |j\rangle$, then what is the state of A? What is the state of B? In terms of the coefficients $\alpha_{ij} \in \mathbb{C}$, when is it possible to write the state of AB as a tensor product of state vectors for the individual systems?

3. Write down a purification for the state $\frac{2}{3}|+\rangle\langle+|_{\mathsf{Q}} + \frac{1}{3}|-\rangle\langle-|_{\mathsf{Q}}$.