4 CHSH game

4.1 The rules of the game

There are two players, Alice and Bob, and a referee. We imagine that Alice and Bob each have their own laboratory. Before the game begins, Alice and Bob can communicate freely. They can discuss their strategy and send each other physical systems. It is just as if they were together in the same laboratory.

During the game all communication between the players is forbidden: There can be no communication whatsoever between Alice's lab and Bob's lab. In fact, they only communicate with the referee, and this communication has the following specific form:

- The referee sends Alice a bit S and Bob a bit T, chosen independently and uniformly at random. That is, for all $s \in \{0,1\}$, $t \in \{0,1\}$, $\Pr(S = s, T = t) = P_{ST}(s,t) = 1/4$.
- Alice must reply to referee with a bit X and Bob with a bit Y.
- They win the game if $X + Y \equiv ST \pmod 2$. We call this event **WIN**.

Note that, for $n, m \in \mathbb{Z}$, $k \in \mathbb{N}$, $k > 0$, $n \equiv m \pmod k$ means $n - m = kj$ for some $j \in \mathbb{Z}$, while $n \bmod k$ means the remainder of n divided by k. (I mixed up these notations in the first lecture, which was confusing.)

4.2 Classical strategies

We make the following assumptions:

1. After the pre-game communication has ceased, but before the game begins, the state of any systems in Alice's lab which she will use to play the game can be described by some random variable A. Similarly the state of Bob's systems can be described by some random variable B.
2. We assume that Alice's answer X is determined by A and the question S she received from the referee, while Bob's answer is determined by T and B. That is,

$$X = f_A(S) \text{ and } Y = g_B(T) \tag{4.1}$$

   where

$$\forall a \in \mathcal{A}_A,\ f_a : \{0,1\} \to \{0,1\} \text{ and} \tag{4.2}$$
$$\forall b \in \mathcal{A}_B,\ g_b : \{0,1\} \to \{0,1\}. \tag{4.3}$$

3. (A, B) is independent of (S, T), i.e.

$$\forall a, b, s, t :\ P_{ST|AB}(s, t|a, b) = P_{ST}(s, t). \tag{4.4}$$

That's it. $\mathcal{A}_A$ and $\mathcal{A}_B$ can be anything you like. In a classical physical model we could take A and B to be points in phase space which represent a snap-shot of the the physical state of the entire contents of the two laboratories before the game begins. Since the players have been communicating freely, A and B could be correlated in any way: we make no assumption on their joint distribution P_{AB}. A model for how X and Y depend on S and T which satisfies these assumptions is called a "local hidden variables model", and A and B are "local hidden variables".

Given these assumptions, what is the maximum possible value of $\Pr(\mathbf{WIN})$? We claim the answer is 3/4. First note that

$$\Pr(\mathbf{WIN}) \leq \max_{a,b} \Pr(\mathbf{WIN}|A=a, B=b) = \Pr(\mathbf{WIN}|A=a^*, B=b^*) \tag{4.5}$$

$$= \sum_{(s,t)\in\{0,1\}^2} \Pr(\mathbf{WIN}, S=s, T=t|A=a^*, B=b^*) \tag{4.6}$$

$$= \sum_{(s,t)\in\{0,1\}^2} \Pr(\mathbf{WIN}|S=s, T=t, A=a^*, B=b^*) P_{ST|AB}(s,t|a^*,b^*) \tag{4.7}$$

$$= \sum_{(s,t)\in\{0,1\}^2} [f(s)+g(t) \equiv st \pmod 2]\frac{1}{4}. \tag{4.8}$$

where $f = f_{a^*}$ and $g = g_{b^*}$ and in (4.8) we used the notation

$$[q] := \begin{cases} 1 \text{ if } q \text{ is true,} \\ 0 \text{ if } q \text{ is false.} \end{cases}$$

Here we are just using basic probability and, in the final equality, the independence of (S,T) and (A,B), (4.4), and $P_{ST}(s,t) = 1/4$, and the rules of the game. The expression (4.8) is the probability that the *deterministic* strategy

$$X = f(S), Y = g(Y)$$

wins the game, so we have shown that there must be an optimal deterministic strategy. Evidently, a deterministic strategy must have $\Pr(\mathbf{WIN}) \in \{0, 1/4, 1/2, 3/4, 1\}$. $\Pr(\mathbf{WIN}) = 3/4$ can be achieved by the strategy both players always answer with zero, since this fails iff $(S,T) = (1,1)$. However, to win with probability one would require that $f(s) + g(t) \equiv 1 \pmod 2$ only for the single pair of questions $(1,1)$. But

$$\sum_{s=0}^{1}\sum_{t=0}^{1}(f(s)+g(t)) = 2\left(\sum_{x=0}^{1} f(s) + \sum_{y=0}^{1} g(t)\right) \equiv 0 \pmod 2,$$

so we must have $f(s) + g(t) \equiv 1 \pmod 2$ for an *even* number m of question pairs $(x,y) \in \{0,1\}^2$. Therefore, the best deterministic strategy has probability 3/4 of winning, and this is maximal among *all* strategies.

4.3 Some quantum mechanics

Here we will review the quantum mechanics that we'll need to describe and analyse a quantum strategy for the CHSH game.

4.3.1 Projectors

Definition 1 (Projectors)**.** $E \in \mathcal{L}(\mathcal{H})$ is a **projector** is $E^\dagger E$. The following statements are equivalent

1. $E^\dagger E = E$.
2. $E^\dagger = E$ and $E^2 = E$.
3. $E^\dagger = E$ and $\mathrm{spec}(E) \subseteq \{0, 1\}$.
4. There is a subspace $S \subseteq \mathcal{H}$ such that, $\forall |\psi\rangle \in \mathcal{H}$, $E|\psi\rangle = |\psi_S\rangle$ where
$$|\psi\rangle = |\psi_S\rangle + |\psi_S^\perp\rangle,$$
is the unique decomposition of $|\psi\rangle$ with $|\psi_S\rangle \in S$ and $|\psi_{S^\perp}\rangle \in S^\perp$ ($S^\perp$ being the orthogonal complement of S in $\mathcal{H}$). We say that E projects onto S.

For finite dimensional $\mathcal{H}$, for any S there is a unique projector E_S which projects onto S. S is the $+1$ eigenspace of E_S and $\dim(S) = \mathrm{rank}(E_S)$.

Given a unit vector $|\psi\rangle \in \mathcal{H}$, the projector onto $\mathrm{span}(\{\psi\})$ is $|\psi\rangle\langle\psi|$. We have the following equivalent ways to represent a state[1] of Q

1. An equivalence class of unit vectors in $\mathcal{H}_\mathsf{Q}$ up to phase: $\{e^{i\phi}|\psi\rangle : \phi \in \mathbb{R}\}$, $\langle\psi|\psi\rangle = 1$.
2. A one-dimensional subspace of $\mathcal{H}_\mathsf{Q}$: $\{ae^{i\phi}|\psi\rangle : \phi, a \in \mathbb{R}\}$.
3. A rank-one projector in $\mathcal{L}(\mathcal{H}_\mathsf{Q})$: $|\psi\rangle\langle\psi|$.

4.3.2 Qubits

If $d_\mathsf{Q} := \dim(\mathcal{H}_\mathsf{Q}) = 2$, then we call Q a **qubit**. (For example, Q is the spin of an electron (or other spin-1/2 particle) or Q is the polarisation of a photon.) In this case, a general state vector $|\psi\rangle$ of Q can be written

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where α and β are any complex numbers satisfying such that $\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$. The corresponding projector is

$$|\psi\rangle\langle\psi| = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}.$$

Note that $\mathrm{Tr}|\psi\rangle\langle\psi| = \langle\psi|\psi\rangle = 1$.

The state vector $\cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$, where $\theta = 2\arccos(|\alpha|) \in [0, \pi]$ and $\phi = \arg(\alpha^*\beta) \in [0, 2\pi)$ is equivalent to $|\psi\rangle$ up to global phase. So, the states of a qubit (according to the state postulate) can be identified with points on the unit sphere by treating (θ, ϕ) as spherical polar coordinates. We call this representation the **Bloch sphere**. Note that antipodal points on the Bloch sphere correspond to pairs of orthogonal states.

[1] According to the state postulate. Soon, we will introduce a more general notion of state.

4.3.3 Composite systems

The composite systems postulate says that, given systems A and B the Hilbert space $\mathcal{H}_{\mathsf{AB}}$ of the composite system AB is $\mathcal{H}_{\mathsf{AB}} = \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$ and if the state of A is $|\psi\rangle_{\mathsf{A}}$ and the state of B is $|\phi\rangle_{\mathsf{B}}$ then the state of AB is $|\psi\rangle_{\mathsf{A}} \otimes |\phi\rangle_{\mathsf{B}}$.

Note that not all state vectors in $\mathcal{H}_{\mathsf{AB}}$ can be written as elementary tensor products, for example

$$|\phi^+\rangle_{\mathsf{AB}} = \frac{1}{\sqrt{2}}(|0\rangle_{\mathsf{A}} \otimes |0\rangle_{\mathsf{B}} + |1\rangle_{\mathsf{A}} \otimes |1\rangle_{\mathsf{B}}).$$

4.3.4 Measurement

Measurement postulate

A measurement on a system Q whose result takes values in $\mathcal{A}$ is represented by a PVM, which is a map $E : \mathcal{A} \to \mathcal{L}(\mathcal{H}_{\mathsf{Q}})$ satisfying

1. $\forall x \in S$, $E(x)$ is a projector;
2. If $x \neq y$ then $E(x)E(y) = 0$;
3. $\sum_{x\in\mathcal{A}} E(x) = \mathbb{1}$ where $\mathbb{1}$ is the identity operator on $\mathcal{H}$.

If X (an RV) is the result of measuring E (so $\mathcal{A}_X = \mathcal{A}$) when the state of Q is $|\psi\rangle$ then

- $\Pr(X = x) = \langle\psi|E(x)|\psi\rangle$;
- Immediately after the measurement, if $X = x$ then the state of Q is $E(x)|\psi\rangle/\|E(x)|\psi\rangle\|$.

Note that $\Pr(X = x) = \mathrm{Tr}|\psi\rangle\langle\psi|E(x) = \langle\psi|E(x)^\dagger E(x)|\psi\rangle = \|E(x)|\psi\rangle\|^2$.

Examples

1. The measurement of an *observable* $M \in \mathrm{Herm}(\mathcal{H}_{\mathsf{Q}})$ with eigenvalues $\mathrm{spec}(M)$, is represented by the PVM E with $\mathcal{A} = \mathrm{spec}(M)$ and $E(x)$ the projector onto the eigenspace $\{|\psi\rangle : M|\psi\rangle = x|\psi\rangle\}$.
2. Given any orthonormal basis $B = \{|e_i\rangle : 1, \dots, d\} \subset \mathcal{H}_{\mathsf{Q}}$, $E : \{1, \dots, d\} \to \mathcal{L}(\mathcal{H}_{\mathsf{Q}}) : i \mapsto |e_i\rangle\langle e_i|$ is a PVM, which we say "measures in the basis B".

4.3.5 Sequential measurements

Note: Unless otherwise stated we are assuming that between measurements there is no time evolution of the systems under consideration i.e. their states do not change. This is the same as assuming a time independent Hamiltonian of the form $H = h\mathbb{1}$ for some $h \in \mathbb{R}$.

1. Suppose that initially the state of Q is $|\psi\rangle$.
2. A PVM $E : \mathcal{A}_X \to \mathcal{L}(\mathcal{H}_{\mathsf{Q}})$ is measured with result X.
3. The measurement postulate says that $P_X(x) = \|E(x)|\psi\rangle\|^2$ and if $X = x$ then the state of Q is $E(x)|\psi\rangle/\sqrt{P_X(x)}$.

4. Now a PVM $F : \mathcal{A}_Y \to \mathcal{L}(\mathcal{H}_\mathsf{Q})$ is measured with result Y.

5. By (3) and the measurement postulate, $P_{Y|X}(y|x) = \langle\psi|E(x)F(y)E(x)|\psi\rangle / P_X(x)$. $P_{YX}(y,x) = P_{Y|X}(y|x)P_X(x) = \langle\psi|E(x)F(y)E(x)|\psi\rangle$.

6. If $X = x$ and $Y = y$, the state of Q is $F(y)E(x)|\psi\rangle/\sqrt{P_{YX}(y,x)}$.

In general $E(x)$ and $F(x)$ may not commute, so the order of the measurements matters.

4.3.6 Measurements on composite systems

Suppose we have a measurement on a system A with PVM $E : \mathcal{A}_X \to \mathcal{L}(\mathcal{H}_\mathsf{A})$. On a composite system AB, this measurement is represented by the PVM $E_\mathsf{A} \otimes \mathbb{1}_\mathsf{B} : x \mapsto E(x)_\mathsf{A} \otimes \mathbb{1}_\mathsf{B} \in \mathcal{L}(\mathcal{H}_\mathsf{A} \otimes \mathcal{H}_\mathsf{B})$. That this must be the case follows from the measurement and composite system postulates. ♣♣Why?

Suppose that this measurement is performed yielding result X, followed by a measurement on B with result Y and PVM $F : \mathcal{A}_Y \to \mathcal{L}(\mathcal{H}_\mathsf{B})$. If the state of AB prior to these measurements is $|\psi\rangle_\mathsf{AB}$ then (see previous section) the joint distribution of the results is

$$\begin{aligned} P_{XY}(x,y) =& \langle\psi|(E(x)_\mathsf{A} \otimes \mathbb{1}_\mathsf{B})(\mathbb{1}_\mathsf{A} \otimes F(y)_\mathsf{B})(E(x)_\mathsf{A} \otimes \mathbb{1}_\mathsf{B})|\psi\rangle && (4.9)\\ =& \langle\psi|(E(x)_\mathsf{A}\mathbb{1}_\mathsf{A}E(x)_\mathsf{A}) \otimes (\mathbb{1}_\mathsf{B}F(y)_\mathsf{B}\mathbb{1}_\mathsf{B})|\psi\rangle && (4.10)\\ =& \langle\psi|E(x)_\mathsf{A} \otimes F(y)_\mathsf{B}|\psi\rangle. && (4.11) \end{aligned}$$

Note that here we would get the same expression for $P_{XY}(x,y)$ if Bob measured before Alice because, for any operators J_A and K_B, $J_\mathsf{A} \otimes \mathbb{1}_\mathsf{B}$ commutes with $\mathbb{1}_\mathsf{A} \otimes K_\mathsf{B}$.

4.4 A quantum strategy for the CHSH game

How might Alice and Bob use quantum systems to play the CHSH game? A simple form of strategy is as follows: Alice has a system A and Bob has a system B. Before the game begins they prepare the composite system AB so that its state is $|\psi\rangle_\mathsf{AB} \in \mathcal{H}_\mathsf{A} \otimes \mathcal{H}_\mathsf{B}$.

Let $E_0 : \{0,1\} \to \mathcal{L}(\mathcal{H}_\mathsf{A})$ and $E_1 : \{0,1\} \to \mathcal{L}(\mathcal{H}_\mathsf{A})$ be PVMs representing measurements of A whose results take values in $\{0,1\}$. In playing the game Alice will use her question bit S to determine which of these measurements she performs, and then use the result of the measurement as her answer, X. That is, Alice measures E_S obtaining a result X. Likewise, Bob measures a PVM G_T on B obtaining a result Y, where $G_0 : \{0,1\} \to \mathcal{L}(\mathcal{H}_\mathsf{B})$ and $G_1 : \{0,1\} \to \mathcal{L}(\mathcal{H}_\mathsf{B})$. From the previous section we know that

$$P_{XY|ST}(x,y|s,t) = \langle\psi|_\mathsf{AB} E_s(x)_\mathsf{A} \otimes G_t(y)_\mathsf{B}|\psi\rangle_\mathsf{AB}. \tag{4.12}$$

The probability that the game is won is

$$\Pr(\mathbf{WIN}) = \sum_{s,t,x,y} [x + y \equiv st \pmod 2] P_{XY|ST}(x,y|s,t)P_{ST}(s,t). \tag{4.13}$$

4.4.1 A particular quantum strategy

Let us take A and B to be qubits (i.e. $d_\mathsf{A} = d_\mathsf{B} = 2$) and let $|\psi\rangle_{\mathsf{AB}} = |\phi^+\rangle_{\mathsf{AB}}$ where

$$|\phi^+\rangle_{\mathsf{AB}} = (|0\rangle_\mathsf{A} \otimes |0\rangle_\mathsf{B} + |1\rangle_\mathsf{A} \otimes |1\rangle_\mathsf{B})/\sqrt{2}. \tag{4.14}$$

Given a qubit Q lets define the state vector $|\eta[\phi]\rangle_\mathsf{Q}$ to be

$$|\eta[\phi]\rangle_\mathsf{Q} := \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)_\mathsf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}e^{i\phi} \end{pmatrix}.$$

This lies on the 'equator' of the Bloch sphere, with azimuthal angle ϕ. Let

$$\eta[\phi]_\mathsf{Q} := |\eta(\phi)\rangle\langle\eta(\phi)|_\mathsf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}e^{i\phi} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}e^{-i\phi} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 1 \end{pmatrix}$$

be the corresponding projector. Note that for all ϕ, $\{|\eta[\phi]\rangle_\mathsf{Q}, |\eta[\phi+\pi]\rangle_\mathsf{Q}\}$ is an orthonormal basis for $\mathcal{H}_\mathsf{Q}$. Therefore, setting

$$\begin{aligned} E_s(x)_\mathsf{A} =& \eta[\pi(s/2 + x)]_\mathsf{A} \text{ and} \\ G_t(y)_\mathsf{B} =& \eta[\pi(t/2 - y - 1/4)]_\mathsf{B} \end{aligned} \tag{4.15}$$

defines valid PVMs. For example, G_0 is a measurement on system B in the basis

$$\{|\eta[-\pi/4]\rangle, |\eta[-3\pi/4]\rangle\}.$$

Now, let's compute $P_{XY|ST}(x, y|s, t)$ for this choice of state and measurements, and then compute the probability of winning the game. First, we note that

$$\begin{aligned} \langle\phi^+|_{\mathsf{AB}}\eta[\alpha]_\mathsf{A} \otimes \eta[\beta]_\mathsf{B}|\phi^+\rangle_{\mathsf{AB}} =& \frac{1}{8}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}^\dagger \begin{pmatrix} 1 & e^{-i\beta} & e^{-i\alpha} & e^{-i(\alpha+\beta)} \\ e^{i\beta} & 1 & e^{i(\beta-\alpha)} & e^{-i\alpha} \\ e^{i\alpha} & e^{i(\alpha-\beta)} & 1 & e^{-i\beta} \\ e^{i(\alpha+\beta)} & e^{i\alpha} & e^{i\beta} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ =& \frac{1}{4}\left(1 + \mathrm{Re}\left(e^{i(\alpha+\beta)}\right)\right). \end{aligned} \tag{4.16}$$

Using (4.12), (4.15) and (4.16), we find

$$\begin{aligned} P_{XY|ST}(x, y|s, t) =& \langle\phi^+|_{\mathsf{AB}}\eta[\pi(s/2 + x)]_\mathsf{A} \otimes \eta[\pi(t/2 - y - 1/4)]_\mathsf{B}|\phi^+\rangle_{\mathsf{AB}} \\ =& \frac{1}{4}\left(1 + \mathrm{Re}\left(e^{i\pi((x-y)+(s+t)/2-1/4)}\right)\right) \end{aligned}$$

Now, since $x, y, s, t \in \{0, 1\}$, we have $e^{i\pi(x-y)} = (-1)^{(x-y)} = (-1)^{(x+y) \bmod 2}$ (which is real) and

$$\mathrm{Re}(e^{i\pi((s+t)/2-1/4)}) = \frac{1}{\sqrt{2}}(-1)^{st}$$

so

$$P_{XY|ST}(x, y|s, t) = \frac{1}{4}\left(1 + \frac{1}{\sqrt{2}}(-1)^{(x+y+st) \bmod 2}\right).$$

The probability that they win the CHSH game, when $(S, T) = (s, t)$ is

$$\Pr(\mathbf{WIN}|S = s, T = t) = \sum_{x,y} [x + y \equiv st \pmod 2] P_{XY|ST}(x, y|s, t), \tag{4.17}$$

$$= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) \tag{4.18}$$

where we used the fact that $x + y \equiv st \pmod 2 \iff (x + y + st) \bmod 2 = 0$, and that there are always exactly two answer pairs (x, y) s.t. $x + y \equiv st \pmod 2$. Therefore,

$$\Pr(\mathbf{WIN}) = \frac{1}{4} \sum_{0 \leq s,t \leq 1} \Pr(\mathbf{WIN}|S = s, T = t) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) \approx 0.854 > 3/4.$$

4.5 Conclusions

The CHSH game can be interpreted as a kind of distributed information processing task. We have shown that using quantum effects allows them perform this simple task with a lower probability of error than would be possible classically. We have also shown that it isn't possible to explain the behaviour of quantum systems by any *local hidden variables* model. That the existence of a local hidden variables model places constraints on the statistics of repeated experiments, which are violated by the predictions of quantum mechanics (and by real experiments! See, for example, `http://fqxi.org/community/forum/topic/2581`), was the insight of John Stewart Bell. The CHSH game is named for the CHSH (Clauser, Horne, Shimony, Holt) inequality, a particular form of constraint which applies to local hidden variable models.